Exercise 2

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx \quad (a > 0).$$

$$Ans.\ \frac{\pi}{2}e^{-a}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \int_{-\infty}^\infty \frac{\cos ax}{2(x^2+1)}\,dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{e^{iaz}}{2(z^2+1)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$2(z^2+1) = 0$$
$$z^2+1 = 0$$
$$z = \pm i$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$.

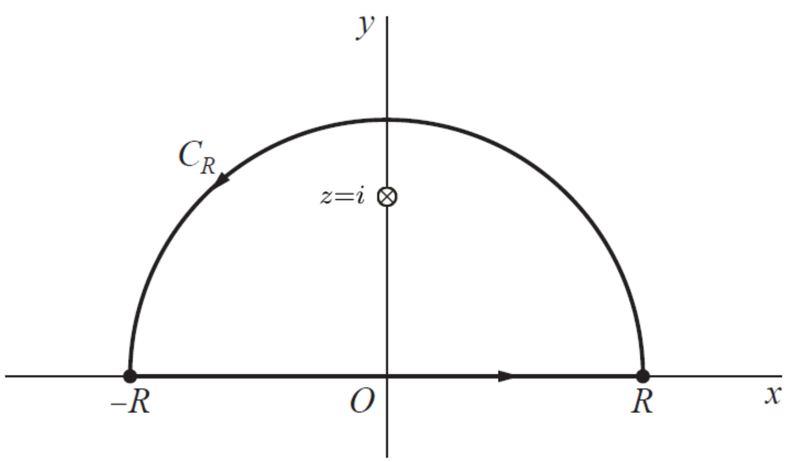

Figure 1: This is Fig. 93 with the singularity at $z = i$ marked.

According to Cauchy's residue theorem, the integral of $e^{iaz}/[2(z^2+1)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{e^{iaz}}{2(z^2+1)}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{iaz}}{2(z^2+1)}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{e^{iaz}}{2(z^2+1)}\,dz + \int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{iaz}}{2(z^2+1)}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, & r = -R \quad &\to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, & \theta = 0 \quad &\to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{e^{iar}}{2(r^2+1)}\,dr + \int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{iaz}}{2(z^2+1)}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{e^{iar}}{2(r^2+1)}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{iaz}}{2(z^2+1)}$$

The denominator can be written as $2(z^2+1) = 2(z+i)(z-i)$. From this we see that the multiplicity of the $z - i$ factor is 1. The residue at $z = i$ can then be calculated by

$$\operatorname*{Res}_{z=i} \frac{e^{iaz}}{2(z^2+1)} = \phi(i),$$

where $\phi(z)$ is equal to $f(z)$ without the $z - i$ factor.

$$\phi(z) = \frac{e^{iaz}}{2(z+i)} \quad \Rightarrow \quad \phi(i) = \frac{e^{i^2 a}}{2(2i)} = \frac{e^{-a}}{4i}$$

So then

$$\operatorname*{Res}_{z=i} \frac{e^{iaz}}{2(z^2+1)} = \frac{e^{-a}}{4i}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{iar}}{2(r^2+1)}\,dr &= 2\pi i \left(\frac{e^{-a}}{4i}\right) \\ \int_{-\infty}^{\infty} \frac{\cos ar + i \sin ar}{2(r^2+1)}\,dr &= \frac{\pi}{2} e^{-a} \\ \int_{-\infty}^{\infty} \frac{\cos ar}{2(r^2+1)}\,dr + i \int_{-\infty}^{\infty} \frac{\sin ar}{2(r^2+1)}\,dr &= \frac{\pi}{2} e^{-a}. \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{\cos ar}{2(r^2+1)}\,dr = \frac{\pi}{2} e^{-a} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin ar}{2(r^2+1)}\,dr = 0$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz &= \int_0^\pi \frac{e^{iaRe^{i\theta}}}{2[(Re^{i\theta})^2+1]}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{e^{iaR(\cos\theta + i\sin\theta)}}{R^2e^{i2\theta}+1}\left(\frac{Rie^{i\theta}}{2}\,d\theta\right) \\ &= \int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^2e^{i2\theta}+1}\left(\frac{Rie^{i\theta}}{2}\,d\theta\right)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz\right| &= \left|\int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^2e^{i2\theta}+1}\left(\frac{Rie^{i\theta}}{2}\,d\theta\right)\right| \\ &\le \int_0^\pi \left|\frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^2e^{i2\theta}+1}\left(\frac{Rie^{i\theta}}{2}\right)\right|d\theta \\ &= \int_0^\pi \frac{\left|e^{iaR\cos\theta}\right|\left|e^{-aR\sin\theta}\right|}{|R^2e^{i2\theta}+1|}\left|\frac{Rie^{i\theta}}{2}\right|d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^2e^{i2\theta}+1|}\frac{R}{2}\,d\theta \\ &\le \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^2e^{i2\theta}|-|1|}\frac{R}{2}\,d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{R^2-1}\frac{R}{2}\,d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{1-\frac{1}{R^2}}\frac{d\theta}{2R}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz\right| \le \lim_{R\to\infty}\int_0^\pi \frac{e^{-aR\sin\theta}}{1-\frac{1}{R^2}}\frac{d\theta}{2R}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz\right| \le \int_0^\pi \lim_{R\to\infty}\frac{e^{-aR\sin\theta}}{1-\frac{1}{R^2}}\frac{d\theta}{2R}$$

Since θ lies between 0 and π, the sine of θ is positive. a is also positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz\right| \le 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz\right| = 0 \quad \to \quad \lim_{R\to\infty}\int_{C_R} \frac{e^{iaz}}{2(z^2+1)}\,dz = 0.$$